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# Accessibility of physical states and non-uniqueness of entanglement measure

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## Abstract

Ordering physical states is the key to quantifying some physical property of the states uniquely. Bipartite pure entangled states are totally ordered under local operations and classical communication (LOCC) in the asymptotic limit and uniquely quantified by the well-known entropy of entanglement. However, we show that mixed entangled states are partially ordered under LOCC even in the asymptotic limit. Therefore, non-uniqueness of entanglement measure is understood on the basis of an *operational* notion of asymptotic convertibility.

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## 1. Introduction

Accessibility between two physical states by some physical process is crucial in being able to compare the states quantitatively. When there exists an *operation* that converts one state into another, we can derive an ordering between the two states from the accessibility based on this operation. This ordering (together with a few other natural assumptions) makes it possible to define a quantity that compares the states. However, if it is impossible to convert one state into another in either direction within a given framework, there exists no coherent way to compare those two states.

Uniqueness of a measure that quantifies a certain physical property is strongly related to ordering of states. When all elements in a given set of physical states can be completely ordered, i.e. an arbitrary two states can be ordered (*total order*), we can make at least one consistent measure that quantifies the set. However, if there exists no ordering that works globally, i.e. a certain pair of states cannot be ordered (*partial order*), then we fail to find a consistent way to ‘align’ all the states. In other words, total order is a necessary condition for a set to be quantified by the unique measure.

One of the most familiar examples in physics that contains partial order is in the special theory of relativity. A pair of events in the spacetime that include each other in their light cone (i.e. the interval between the two events is time-like) are accessible because one can affect the other by sending some signal. However, if one is outside the light cone of the other (i.e. the interval between the two events is space-like), then it is impossible to connect them by any physical operation. Therefore, there exists no unique way of ordering two such states; different orderings are possible by choosing different reference frames. Therefore, the set of events is a partially ordered one, which leads to the well-known non-uniqueness of simultaneity that follows from the principles of special theory of relativity (see chapter 17 of [1], for example). Furthermore, in a modern approach to relativity, a fundamental structure of spacetime is modelled as a partially ordered set called a causal set [2].

An important example of partial order in a purely mathematical context is the theory of majorization [3], which is a powerful tool for comparing two vectors and deriving various inequalities between operators. For example, majorization brings partial order to probability vectors and leads to useful inequalities of quantities related to entropy in statistical mechanics [4]. An intimate relation between majorization and quantum information theory has also been discovered recently, which we will mention below.

The most beautiful and successful application of the theory of ordering physical states is in thermodynamics, where all equilibrium states are totally ordered under adiabatic processes and quantified by the unique measure of entropy. Given two equilibrium states (A and B), entropy  $S$  distinguishes between possible and impossible directions of adiabatic processes between the two states; A can access B via an adiabatic process iff  $S(A) \leq S(B)$ . (If the equality holds, B can also access A and so the process becomes reversible.) The uniqueness of entropy was proved by Giles with his axiomatic approach, which was developed to clarify the structure of thermodynamics [5]. Giles's work is a culmination of the movement towards a more lucid understanding of the second law of thermodynamics, starting with Carathéodory (see [6]). This approach has recently been revisited by Lieb and Yngvason [7, 8].

It has been shown recently that thermodynamics and the theory of pure-state entanglement share the same mathematical structure, Giles's rigorous set of mathematical axioms. Adiabatic processes in thermodynamics correspond to manipulations of bipartite pure entangled states by local operations and classical communication (LOCC) in the context of quantum information theory [9]. Therefore, bipartite pure entangled states are totally ordered under LOCC in the asymptotic limit, and entropy gives the unique measure in this context as well (known as the von Neumann entropy of entanglement [10, 11]).

In quantum information theory [12], quantum entanglement has been a subject of intensive research because it is a new resource in physics as well as an indispensable resource in quantum information processing. As in the case of other physical resources, it is desirable to find a unique measure of entanglement in order to exploit it effectively and efficiently. (For a review of entanglement measures see [13].) In contrast to the case of bipartite pure states, the unique measure of entanglement in mixed states has not yet been established. It has been proved that if two entanglement measures coincide in pure states but differ in mixed states, then they impose different orderings [14]. In fact, some entanglement measures proposed so far are different, and it is commonly believed that we need different entanglement measures depending on the scenario.

In this paper, we show that mixed entangled states are partially ordered under an *operational* notion of asymptotic LOCC convertibility by using the monotonicity of entanglement cost and the fact that positive partial transposition (PPT) bound entangled states cannot be converted into negative-partial-transposition (NPT) entangled states by LOCC. Thus, we point out that the partial order structure underlies the non-uniqueness of

entanglement measure. This immediately reveals the reason why Giles’s axiom fails in mixed-state entanglement, especially axiom 5, which reads *if two states A and B are both accessible from another state C, then A and B are accessible in either direction (or both)*. This is exactly what distinguishes total order from partial order. We show the partially ordered structure of mixed entangled states by giving a counterexample to axiom 5. (For other natural axioms and details of Giles’s approach, see [5, 9].)

The violation of axiom 5 can be seen, for example, in the theory of relativity mentioned above: even if two events A and B are accessible from another event C, the events A and B are not necessarily accessible from each other because one can be outside the light cone of the other. Another example violating axiom 5 is in entanglement manipulation of bipartite pure states in *finite* regimes. For example, although both  $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  and  $|\phi_2\rangle = \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{6}}|11\rangle + \frac{1}{\sqrt{6}}|22\rangle$  can be obtained from a maximally entangled state  $|\Phi_3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$  by LOCC,  $|\phi_1\rangle$  and  $|\phi_2\rangle$  cannot be converted into each other by LOCC with certainty. This is a direct consequence of Nielsen’s theorem, which connects entanglement manipulation and majorization mentioned above [15]. In the following, we will say two states A and B are *incomparable* if they are not accessible from each other.

The rest of this paper is organized as follows. First, we present a counterexample, and thus prove the partial order structure of mixed entangled states. Then, we discuss one possible way of recovering total order. Finally, we conclude the paper with future directions.

## 2. Partial order on mixed entangled states

First, let us rigorously define the accessibility in axiom 5 within our context. The asymptotic convertibility under LOCC is defined as follows: a state  $\rho$  is convertible into a state  $\sigma$  if and only if for every (arbitrarily small) real number  $\epsilon$ , there exists an integer  $n_0$ , and a sequence of LOCC  $L_n$  such that for any integer  $n \geq n_0$  we have

$$\|L_n(\rho^{\otimes n}) - \sigma^{\otimes n}\| \leq \epsilon \tag{1}$$

where  $\rho^{\otimes n} = \rho \otimes \rho \otimes \dots \otimes \rho$  represents a tensor product of  $n$  copies of the state  $\rho$  and  $\|\dots\|$  denotes the usual trace norm distance between two mixed quantum states. Loosely speaking, this means that one state can be converted into another if a certain number of copies of the former can arrive at an arbitrarily good approximation of the *same* number of copies of the latter via LOCC in the asymptotic limit. We will prove that the set of mixed entangled states is a partially ordered one under this definition of accessibility. This is our central result in this paper. While partial order in bipartite pure states in finite regimes can be turned into total order in the asymptotic limit under this accessibility, mixed entangled states still retain partial order structure even in the asymptotic limit. Note that we consider the convertibility between the *same* number of copies here unlike the ordinary argument of transformations between different numbers of copies, e.g., from  $n$  copies to  $m$  copies. In the transformations between different numbers of copies, it is no wonder that the transformation is possible at least in one direction if sufficiently large number of copies of the initial state are prepared. Thus, the partial order structure does not clearly appear in the framework, while it can be easily seen in transformations with the same number of copies as we will show below. It is enough to consider the convertibility between the same number of copies to see whether it is possible to compare them because if one state is ‘larger’ than the other, the former should be converted into the latter even in this framework.

Intuitively, the bipartite mixed states that are most likely to fail axiom 5 are bound entangled states [16]. Since bound entangled states are mixed states from which no entangled

pure state can be distilled, if we take one of those,  $\rho_{AB}$ , and a pure entangled state,  $\sigma_{AB}$ , as a pair of possible candidates for a counterexample, the first half of the proof has already been accomplished by definition, i.e.  $\rho_{AB} \rightarrow \sigma_{AB}$  is impossible for all pure states  $\sigma_{AB}$ . So, all we have to do is to disprove the convertibility in the opposite direction.

In order to prove that, we take a particular bound entangled state constructed from an unextendible product basis (UPB) [17]. Suppose both Alice and Bob have three-level quantum systems (qutrits). Consider the following incomplete orthonormal product basis:

$$\begin{aligned} |\psi_1\rangle &= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |2\rangle \\ |\psi_3\rangle &= |2\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \\ |\psi_4\rangle &= (|1\rangle + |2\rangle) \otimes |0\rangle \\ |\psi_5\rangle &= \frac{1}{\sqrt{3}}(|0\rangle - |1\rangle + |2\rangle) \otimes \frac{1}{\sqrt{3}}(|0\rangle - |1\rangle + |2\rangle). \end{aligned} \quad (2)$$

This incomplete orthogonal basis forms a UPB, which means that there exists no product state orthogonal to all of the above five states. Consequently, the four-dimensional subspace complementary to this five-dimensional one does not contain any product states. Therefore, with a normalization factor, the projection operator onto this complementary space

$$\rho_{AB} = \frac{1}{4} \left( I - \sum_{i=1}^5 |\psi_i\rangle\langle\psi_i| \right) \quad (3)$$

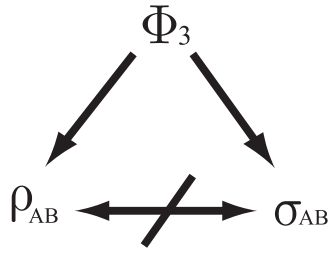
turns out to be an entangled state. It can also be easily seen that this state satisfies the PPT condition because the identity operator and projections onto product states like  $|\psi_i\rangle\langle\psi_i|$  remain positive after partial transposition. Thus,  $\rho_{AB}$  is proved to be a PPT bound entangled state.

The important fact about the state  $\rho_{AB}$  is that its entanglement cost  $E_C(\rho_{AB})$  is positive [18], which is defined as  $E_C(\rho) \equiv \lim_{n \rightarrow \infty} E_f(\rho^{\otimes n})/n$  [19], where  $E_f(\rho)$  represents the entanglement of formation of  $\rho$  [20]. (It is obvious that some amount of entanglement is necessary to construct one copy of a bound entangled state. Until quite recently, however, it was an open question whether the entanglement cost of bound entangled states is also positive [18].) Owing to this property, one can choose a pure entangled state  $\sigma_{AB} = |\phi\rangle\langle\phi|$  such that

$$0 < E_C(\sigma_{AB}) < E_C(\rho_{AB}). \quad (4)$$

For simplicity, we choose  $|\phi\rangle$  to be an entangled states with Schmidt number two or three, i.e. a  $2 \times 2$  or  $3 \times 3$  system. Since the entanglement cost  $E_C$  is an entanglement monotone, i.e. it cannot increase under LOCC,  $\sigma_{AB}$  can never be converted into  $\rho_{AB}$  even asymptotically, i.e.  $\sigma_{AB} \rightarrow \rho_{AB}$  is impossible. The monotonicity of entanglement cost  $E_C$  can be easily derived from the fact that entanglement of formation  $E_f$  is also an entanglement monotone. Note that the above incomparability holds in the sense of the *same* number of copies. Otherwise, a sufficiently large number of copies of  $\sigma_{AB}$  can always produce a much smaller number of copies of  $\rho_{AB}$  with certainty.

Besides the above fact, note that a maximally entangled state  $|\Phi_3\rangle_{AB}$  can access both  $\rho_{AB}$  and  $\sigma_{AB}$ . (Although the state  $|\Phi_3\rangle_{AB}$  might not be the most efficient example to produce them, this does not matter here. One can make one copy of either  $\rho_{AB}$  or  $\sigma_{AB}$  from one copy of  $|\Phi_3\rangle_{AB}$  by LOCC.) Therefore, we found a counterexample that the two states  $\rho_{AB}$  and  $\sigma_{AB}$  are *not* convertible into each other in spite of the fact that both of them can be accessed from the same state  $|\Phi_3\rangle_{AB}$ . This clearly shows that entangled mixed states are partially ordered under LOCC in the asymptotic limit and thus violate axiom 5. (We note that although, in Giles's axioms, transformations assisted by asymptotically negligible amount of auxiliary states are



**Figure 1.** Partial order structure in mixed entangled states.  $|\Phi_3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ ,  $\rho_{AB}$  is a bound entangled state defined in equation (3), and  $\sigma_{AB}$  is a pure entangled state satisfying  $0 < E_C(\sigma_{AB}) < E_C(\rho_{AB})$ . The pure state  $\sigma_{AB}$  can be replaced with any NPT entangled state  $\chi_{AB}$  satisfying the same condition.

considered, the undistillable property of bound entanglement remains unchanged even with the assistance of auxiliary entangled states [18].)

In the above argument, we chose a pure state as  $\sigma_{AB}$  for simplicity. However, it can be replaced with distillable mixed states  $\chi_{AB}$  such that  $0 < E_C(\chi_{AB}) < E_C(\rho_{AB})$  because PPT bound entangled states cannot be converted into NPT ones by LOCC, i.e.  $\rho_{AB} \rightarrow \chi_{AB}$  is impossible for all NPT states  $\chi_{AB}$ . Thus, the above also holds for any such  $\chi_{AB}$ . (Furthermore, the monotonicity of other entanglement measures can also be used for the above argument instead of entanglement cost.) Generally, it is concluded that any PPT bound entangled state with positive entanglement cost always has incomparable states in the NPT regime (see figure 1). These states are the counterexamples to axiom 5, which prevents us from applying Giles's approach to mixed entangled states within the framework of LOCC convertibility.

Therefore, we have proved that the set of mixed entangled states is a partially ordered one under the operational notion of asymptotic convertibility with LOCC, which underlies the non-uniqueness of entanglement measure. Since there is no *operational* way to link incomparable states, there exists no way of assigning meaningful amounts of entanglement to them that could determine which state is more entangled. In other words, it is the partial order that allows us to use various entanglement measures known so far without any contradiction. Metaphorically speaking, we have shown a sort of 'relativity' of entanglement measure under asymptotic convertibility with LOCC, which means that there exists no *absolute* entanglement measure at least under LOCC.

### 3. Recovering total order

Next we discuss the possibility of recovering total order from partial order, which might lead to a unique measure of entanglement. Obviously, an extra operation besides LOCC will be necessary to achieve it. One naive way is restoring quantum information discarded into the environment, which can be seen as follows.

First, let us think about how the total order structure of pure entangled states changes into a partial one during the formation process of mixed entangled states. Imagine a process making the state  $\rho_{AB}$  in equation (3) from a maximally entangled state by LOCC. The state can be written as  $\rho_{AB} = \frac{1}{4} \sum_{i=6}^9 |\psi_i\rangle\langle\psi_i|$ , where  $|\psi_i\rangle$  ( $i = 6, \dots, 9$ ) is an entangled basis complementary to the UPB in equation (2). One way of making  $\rho_{AB}$  is as follows (it is not necessarily the most efficient way from the viewpoint of the amount of entanglement invested). Alice and Bob first prepare a three-level maximally entangled state  $|\Phi_3\rangle_{DB}$  between them. Besides this, Alice prepares two qutrits, A, B', and a four-level ancilla C (e.g. two

qubits) locally, with which she makes a superposed state  $|\omega\rangle_{CAB'} = \frac{1}{2} \sum_{i=6}^9 |i\rangle_C |\psi_i\rangle_{AB'}$ . By teleporting the system B' to Bob with the previously shared entanglement  $|\Phi_3\rangle_{DB}$ , they succeed in constructing a state  $|\psi\rangle_{CAB} = \frac{1}{2} \sum_{i=6}^9 |i\rangle_C |\psi_i\rangle_{AB}$  between them. This state changes into the mixed state  $\rho_{AB}$  immediately after Alice throws away the system C into the environment.

Generally, the total order structure survives until Alice discards the information because the entire state is just a pure state. Therefore, if she could retrieve the quantum information from the environment, the total order would be recovered as well. Although this may appear impractical if we try to find *realistic* operations that make it possible to restore quantum information from the environment, it is possible in principle. In fact, we can also imagine some artificial situation, which might be plausible in the context of information processing, where Alice gives the system C to the third party, e.g. Charlie, not to the environment. In this situation, restoring the quantum information is not impractical at all. In other words, if we purify (mathematically) a bipartite mixed state  $\rho_{AB}$  into a pure state  $|\psi\rangle_{CAB}$  by introducing a local ancilla C at Alice's side *virtually*, then all states become pure states and we should recover total order.

With this restoring process, we can tell which state of incomparable states would have more entanglement than the other, if Alice had not lost the quantum information in the formation process. We quantify entanglement of pure states by using entropy of entanglement as usual: if the restored state is  $|\psi\rangle_{CAB}$ , then we define the amount of entanglement  $E(\rho_{AB})$  between Alice (CA) and Bob (B) as the von Neumann entropy of Bob's reduced density matrix  $\text{tr}_{CA}(|\psi\rangle_{CAB}\langle\psi|)$ . It is shown below that the bound entangled state  $\rho_{AB}$  would have more entanglement than the incomparable state  $\sigma_{AB} = |\phi\rangle\langle\phi|$ . (The approach here is similar to the definition of entanglement of purification [21], where the entanglement is minimized over all possible purification on both Alice's and Bob's sides. Thus, our quantity is always greater than this.)

It is easily seen that  $E(\rho_{AB}^{\otimes n}) \geq E_f(\rho_{AB}^{\otimes n})$  due to the concavity of the von Neumann entropy [20]. Thus, we also have  $\lim_{n \rightarrow \infty} E(\rho_{AB}^{\otimes n})/n = E(\rho_{AB}) \geq E_C(\rho_{AB}) = \lim_{n \rightarrow \infty} E_f(\rho_{AB}^{\otimes n})/n$ . Note that  $E$  and  $E_C$  coincide for pure states. Since we chose  $\sigma_{AB}$  according to equation (4),  $E$  of the bound entangled state is always greater than that of the incomparable pure state, i.e.  $E(\sigma_{AB}) = E_C(\sigma_{AB}) < E_C(\rho_{AB}) \leq E(\rho_{AB})$ . Therefore, any incomparable states with equation (4) satisfy  $E(\sigma_{AB}) < E(\rho_{AB})$ .

We have considered a simple example of recovering total order by restoring information from the environment. Besides the above case, we can conceive other scenarios with different formation processes. For example, Alice could perform a measurement on the system C and then discard the classical information of the outcome instead of discarding the quantum system directly. In this case, in order to recover total order, she would restore just classical information and generally end up in a different pure state from the original one. A striking example is  $\frac{1}{2}(|00\rangle_{AB}\langle 00| + |11\rangle_{AB}\langle 11|)$ , which will be recovered to a GHZ state in the former scenario, while it will be recovered to just a separable state in the latter. (In fact, the above quantification of  $E$  assigns a positive value 1 to this state. This unpleasant quantification is due to the prescription designed only to compare incomparable states.) Therefore, it is fair to say that recovered total order is highly dependent on how information was lost and retrieved. (See also [22] in a different context of entanglement versus information loss.)

#### 4. Discussion and conclusion

It is worth investigating the partial order structure of mixed entangled states in detail. Although it was proved only for bound entangled states, we believe the partial order structure is embedded in distillable states as well. It is also known that there exist distillable states for which



entanglement cost is strictly greater than distillable entanglement [23, 24]. Thus, this gap between entanglement cost and distillable entanglement also leads to the same argument as in the case of bound entangled states just proved: suppose  $\rho_{AB}$  is a distillable mixed state such that  $E_D(\rho_{AB}) < E_C(\rho_{AB})$ . Then,  $\rho_{AB}$  is incomparable with a pure state whose entropy of entanglement is sandwiched between  $E_D(\rho_{AB})$  and  $E_C(\rho_{AB})$ . (The monotonicity of distillable entanglement is invoked here instead of the fact that PPT bound entangled states are not accessible to NPT entangled states.) Therefore, it is concluded that the partial order structure lies in distillable states as well.

In summary, we proved the partial order structure of mixed entangled states under the operational notion of asymptotic convertibility with LOCC, which makes it possible to understand the non-uniqueness of entanglement measure based on a very general argument on physical accessibility. The origin of the partial order is likely to be closely related to information loss. An important future direction is finding out exactly how the thermodynamical structure of pure states breaks down when mixedness appears in entanglement.

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